# Math 249, Wed. April 1

$S_4$

| | $(1^4)$ | $(211)$ | $(22)$ | $(31)$ | $(4)$ |
|---|---|---|---|---|---|
| $\chi_{(4)}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_{(31)}$ | 3 | 1 | -1 | 0 | -1 |
| $\chi_{(22)}$ | 2 | 0 | 2 | -1 | 0 |
| $\varepsilon \otimes \chi_{(31)} = \chi_{(211)}$ | 3 | -1 | -1 | 0 | 1 |
| $\chi_{(1111)}$ | 1 | -1 | 1 | 1 | -1 |

$\chi_\lambda \longleftrightarrow s_\lambda$

$\chi_\lambda(w_\tau) = \langle s_\lambda, p_\tau \rangle$

$\chi_{(n)} \qquad s_{(n)} = h_n$

$\chi_{(1^n)} \qquad s_{(1^n)} = e_n$

$\dim \chi_\lambda = |SYT(\lambda)|$

$(31)$

$\begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix} \qquad \begin{matrix} 2 & 4 \\ 1 & 3 \end{matrix}$

What is $\chi_{(22)}$?

$1 \leftrightarrow 2$

$3 \quad 4$

$\to \mathbb{C}^4 = \mathbb{C} \oplus \underbrace{V}_{\dim 3} \leftarrow \chi_{(4)} + \chi_{(31)}$

$\uparrow$

$S_4 / S_3 \times S_1$

$= \operatorname{Ind}_{S_3 \times S_1}^{S_4}(1) \xrightarrow{F} h_{31} = s_4 + s_{31}$

$\chi_{(31)}$

$S_4 \twoheadrightarrow S_3$

$S_3 \curvearrowright \mathbb{C}^3 = \mathbb{C} \oplus \underset{\dim 2}{V}$

$S_4$ acting on matchings on $[4]$

$1 \leftrightarrow 2$

$3 \leftrightarrow 4$

$\langle s_\lambda, p_\tau \rangle$

Murnaghan-Nakayama Rule for $\chi_\lambda(w_\tau) = \langle s_\lambda, p_\tau \rangle$.

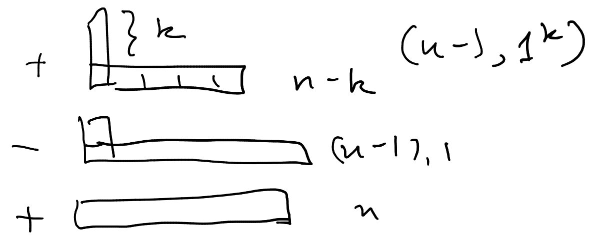
Lemma $p_n = s_{(n)} - s_{(n-1,1)} + s_{(n-2,1^2)} - \cdots (-1)^{n-1} s_{(1^n)}$

Proof $p_n a_\delta$ $\quad x_1^{l-1} x_2^{l-2} \cdots x_l^0 \cdot x_i^n$

$\delta + ne_1 = (l-1, l-2, \dots, 0)$

$= (n+l-1, l-2, \dots, 0) = (n, 0 \dots 0) + \delta$

$a_{\lambda+\delta}$ $\quad \lambda = (n)$




$s_\lambda = a_{\lambda+\delta} / a_\delta$

$p_n a_\delta \cdots$

$\delta + ne_2 = (l-1, n+l-2, l-3, \dots, 0)$

$(n+l-2, l-1, l-3, \dots, 0)$

$-a_{\lambda+\delta}$ $\quad \lambda = (n-1, 1, 0, \dots)$

$\delta + ne_k = (l-1, \dots, l-k+1, n+l-k, l-k-1, \dots, 0) \to 0$ if $n < k$

$\approx (n+l-k, l-1, \dots, l-k+1, l-k-1, \dots, 0)$

$(-1)^{k-1} a_{\lambda+\delta}$ $\quad \lambda = (n-k+1, 1^{k-1})$

$p_n a_\delta = a_{(n)+\delta} - a_{(n-1,1)+\delta} + a_{(n-2,1^2)+\delta} - \cdots \pm a_{(1^n)+\delta}$

$p_n = s_{(n)} - s_{(n-1,1)} + s_{(n-2,1^2)} \cdots$

Cor. $p_n = \left.\dfrac{h_n[X(1-u)]}{1-u}\right|_{u=1}$ $\qquad (1-u)p_n = h_n[X(1-u)] = h_n[X - uX]$

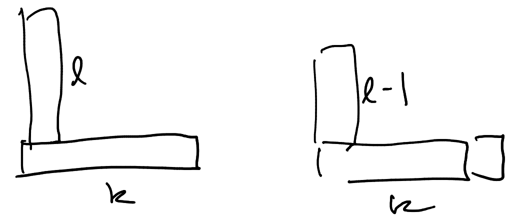
$h_n[X+Y] = \sum_{k+l=n} h_k(X) h_l(Y)$ $\qquad \Omega[X+Y] = \Omega[X]\Omega[Y]$

$$h_n[X-uX] = \sum_{k+l=n} h_k(X)\, \underbrace{h_l[-uX]}$$

$$u^l h_l[-X] = u^l(-1)^l\, \omega h_l(X) = (-u)^l e_l(X)$$

$$h_n[X-uX] = \sum_{k+l=n} (-u)^l h_k e_l$$

$$h_k e_l = S_{(k)} S_{(1^l)} = S_{(k,1^l)} + S_{(k+1,1^{l-1})}$$




$$S_{(n)} - u\left(S_{(n)} + S_{(n-1,1)}\right) + u^2\left(S_{(n-1,1)} + S_{(n-2,2)}\right) - u^3 \cdots (-u)^n\left(S_{(1^l)}\right)$$
$$+ (-u)^{n-1}\left(S_{(2,1^{l-2})} + S_{(1^l)}\right)$$

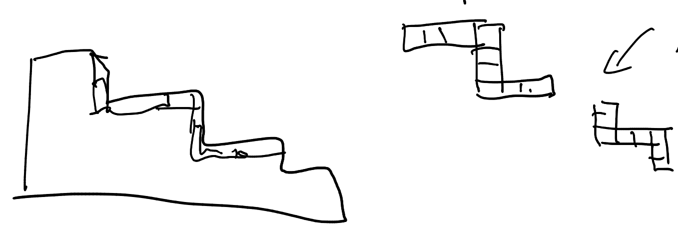
$$(1-u)S_{(n)} - u(1-u)S_{(n-1,1)} \cdots$$
$$= \cancel{(1-u)}\left(S_{(n)} - uS_{(n-1,1)} + u^2 S_{(n-2,1^2)} \cdots\right)$$

$$\left.\frac{h_n[X(1-u)]}{1-u}\right|_{u=1} = S_{(n)} - S_{(n-1,1)} + \cdots = p_n$$

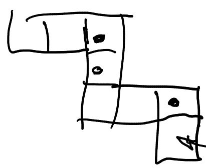
$$\langle S_\lambda, p_\tau\rangle = \langle p_{\tau_1}^\perp S_\lambda, p_{\tau_2}\cdots p_{\tau_k}\rangle = \langle p_{\tau_1}^\perp p_{\tau_2}^\perp S_\lambda, p_{\tau_3}\cdots p_{\tau_k}\rangle$$
$$= \langle p_\tau^\perp S_\lambda, 1\rangle = p_\tau^\perp S_\lambda$$

$$h_k[X(1-u)]^\perp S_\lambda$$
$$\sum_{r+s=k} h_r(X)^\perp e_s(X)^\perp (-u)^l$$

$S_\lambda \mapsto$



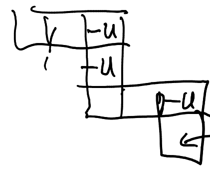
Ribbons



$(-u)^3$ $\leftarrow (\circ)$

$k=5 \quad l=3$

$k=4 \quad l=4 \quad (-u)^4$



$-u$, $-u$, $-u$, $\leftarrow 1-u$

$(-u)^3(1-u)$

$$h_k[X(1-u)]^{\perp} s_\lambda$$

$$\| $$

$$\sum s_\mu \cdot (1-u)^{\#\text{components}} (-u)^{\#\text{rows} - \#\text{components}}$$

$\lambda \setminus \mu$ ribbon

$$p_k^{\perp} s_\lambda = \sum s_\mu \, (-1)^{\#\text{rows}-1}$$

$\lambda \setminus \mu$ connected ribbon of size $k$

$p_\tau^{\perp} s_\lambda$ count <u>connected ribbon tableaux</u> of weight $\tau$ with signs $\prod (-1)^{\#\text{rows}-1}$

$p_{431}^{\perp}$

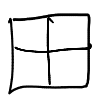
$$\begin{array}{llll} 2 & & & \\ 2 & 2 & 3 & \\ 1 & 1 & 1 & 1 \end{array}$$

$(-1)$

$(+1)(-1)(+1)$

$$\langle s_\lambda, p_1^n \rangle = |SYT(\lambda)|$$

$\chi_{22}$ $\quad \langle s_{22}, p_\tau \rangle$ $\quad \tau = 1^4 \quad \tau = 211 \quad \tau = (22) \quad \tau = (31) \quad \tau = (4)$



| | $\tau = 1^4$ | $\tau = 211$ | $\tau = (22)$ | $\tau = (31)$ | $\tau = (4)$ |
|---|---|---|---|---|---|
| | $+1$ $\begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array}$ | $\begin{array}{ll} 2 & 3 \\ 1 & 1 \end{array}$ $+1$ | $\begin{array}{ll} 2 & 2 \\ 1 & 1 \end{array}$ $+1$ | $\begin{array}{ll} 1 & 2 \\ 1 & 1 \end{array}$ $-1$ | |
| | $+1$ $\begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}$ | $\begin{array}{ll} 1 & 3 \\ 1 & 2 \end{array}$ $-1$ | $\begin{array}{ll} 1 & 2 \\ 1 & 2 \end{array}$ $+1$ | | |
| | $2$ | $0$ | $2$ | $-1$ | $0$ |

$\chi_{\boxplus} \quad 2 \quad 0 \quad 2 \quad -1 \quad 0$

$s_\lambda \cdot s_\mu$ $\quad |\lambda| = k \quad |\mu| = \ell \quad S_k \quad S_\ell$

$$\overset{\shortparallel}{\sum_\nu} \underbrace{c^\nu_{\lambda\mu}}_{\text{LR coefficients}} s_\nu \qquad |\nu| = n = k+\ell \qquad S_n$$

$V \otimes W$

Induction Product $\quad \chi \in X(S_k) \quad \varphi \in X(S_\ell) \qquad \chi \otimes \varphi$ is an $S_k \times S_\ell$ character

$X(S_n) \ni \quad \chi * \varphi = \mathrm{Ind}_{S_k \times S_\ell}^{S_n}(\chi \otimes \varphi) \qquad (\chi \otimes \varphi)(g,h) = \chi(g)\varphi(h)$

$n = k+\ell$

$$\mathrm{Ind}_{S_\lambda}^{S_k}(1) * \mathrm{Ind}_{S_\mu}^{S_\ell}(1) = \mathrm{Ind}_{S_{(\lambda;\mu)}}^{S_n} \qquad \mathrm{Ind}_{S_k \times S_\ell}^{S_n}\left(\mathrm{Ind}_{S_\lambda \times S_\mu}^{S_k \times S_\ell}(1)\right)$$

$F \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad = \mathrm{Ind}_{S_\lambda \times S_\mu}^{S_n}(1)$

$h_\lambda \qquad h_\mu \qquad h_{\lambda;\mu} = h_\lambda h_\mu \qquad S_{(\lambda;\mu)}$

$$\Rightarrow F(\chi * \varphi) = F(\chi) F(\varphi)$$

$$F : \bigoplus_n X(S_n) \xrightarrow[\text{ring iso}]{} \Lambda_{\mathbb{Z}}$$

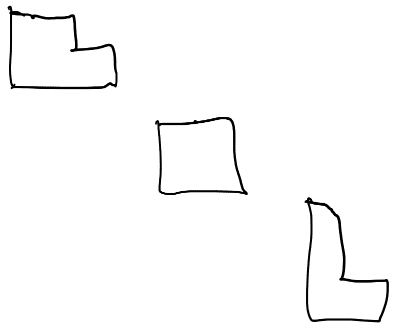
$\Rightarrow c^\nu_{\lambda\mu} = (s_\nu, s_\lambda \cdot s_\mu) \qquad s_\lambda s_\mu = F(\chi_\lambda) F(\chi_\mu)$

$=$ multiplicity of $\chi_\nu$ in $\chi_\lambda * \chi_\mu \geq 0 \qquad = F(\underset{\uparrow \text{ actual char}}{\chi_\lambda * \chi_\mu})$ \$

Combinatorial question: What does $c^\nu_{\lambda\mu}$ count?

$c^{\nu}_{\lambda^{(1)},\dots,\lambda^{(k)}} = \langle s_\nu , s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(k)}} \rangle$ also ?

Notice that $s_{\lambda^{(1)}} \cdots s_{\lambda^{(k)}}$ is a skew schur function $s_{\lambda^{(1)} \oplus \cdots \oplus \lambda^{(k)}}$



$$c^{\nu}_{\lambda^{(1)},\dots,\lambda^{(k)}} = \langle s_\nu , s_{\lambda/\mu} \rangle \qquad \lambda/\mu = \lambda^{(1)} \oplus \cdots \oplus \lambda^{(k)}$$

$$\| $$

$$\langle s_\nu s_\mu , s_\lambda \rangle = c^{\lambda}_{\mu\nu}$$

$$K_{\lambda\mu} = |SSYT(\lambda,\mu)| = \langle m_\mu \rangle s_\lambda = \langle h_\mu , s_\lambda \rangle = \langle s_\lambda \rangle h_\mu = \langle s_\lambda \rangle s_{(\mu_1)} \cdots s_{(\mu_k)}$$

$$= c^{\lambda}_{(\mu_1),\dots,(\mu_k)}$$